ON CYCLICALLY SYMMETRICAL SPACETIMES

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In a recent paper Carot et al. considered the definition of cylindrical symmetry as a specialisation of the case of axial symmetry. One of their propositions states that if there is a second Killing vector, which together with the one generating the axial symmetry, forms the basis of a two-dimensional Lie algebra, then the two Killing vectors must commute, thus generating an Abelian group. In this paper a similar result, valid under considerably weaker assumptions, is derived: any two-dimensional Lie transformation group which contains a one-dimensional subgroup whose orbits are circles, must be Abelian. The method used to prove this result is extended to apply to three-dimensional Lie transformation groups. It is shown that the existence of a one-dimensional subgroup with closed orbits restricts the Bianchi type of the associated Lie algebra to be I, II, III, $\mathrm{VII}_{q=0}$, VIII or IX. Some results on n-dimensional Lie groups are also derived and applied to show there are severe restrictions on the structure of the allowed four-dimensional Lie transformation groups compatible with cyclic symmetry.

1 Introduction

Following Carter [1] a spacetime $\mathcal{M}$ is said to have cyclical symmetry if and only if the metric is invariant under the effective smooth action $SO(2) \times \mathcal{M} \to \mathcal{M}$ of the one-parameter cyclic group $SO(2)$. A cyclically symmetric spacetime in which the set of fixed points of this isometry is not empty is said to be axially symmetric and the set of fixed points itself is referred to as the axis (of symmetry). Mars and Senovilla [2] proved a number of useful results on the structure of the axis. Carot, Senovilla and Vera [3] considered a definition of cylindrical symmetry based on the following proposition: if in an axial symmetric spacetime there is a second Killing vector which, with the Killing vector generating the axial symmetry, generates a two-dimensional isometry group then the two Killing vectors commute and the isometry group is Abelian. A similar result for stationary axistymmetric spacetimes was proved by Carter [1].

The proofs of all the above mentioned results rely heavily on the existence of an axis and although the assumption of the existence of an axis is reasonable in many circumstances, there are numerous situations where an axis in a cyclically symmetric manifold may not exist. The 'axis' may be singular due to line sources and so not part of the manifold proper or the topology of the manifold may be such that no axis exists as is the case for the standard two-dimensional torus embedded in three-dimensional Euclidean space. In the next section the

condition for the existence of an axis will be discarded and the following result will be proved: any two-dimensional Lie transformation group which acts on an n-dimensional manifold $\mathcal{M}$ and which contains a one-dimensional subgroup acting cyclically on $\mathcal{M}$ must be Abelian. In subsequent sections three and higher dimensional Lie transformation groups will be considered and their structure shown to be severely restricted by the existence of a one-dimensional subgroup with circular orbits.

2 Cyclically Symmetric Manifolds Admitting a G_2

Suppose X_0 is the Killing vector field associated with the cyclic isometry acting on $\mathcal{M}$ and let $\mathcal{N}$ be the open submanifold of $\mathcal{M}$ on which $X_0 \neq 0$. The orbit of each point of $\mathcal{N}$ under the cyclic symmetry is a circle. Let ϕ be a circular coordinate running from 0 to 2π which parameterises elements of $SO(2)$ in the normal way. Then we can introduce a coordinate system x^i with $i = 1 \ldots n$ and $x^1 = \phi$ adapted to X_0 such that $X_0 = \partial_\phi$.

Suppose that the isometry group of $\mathcal{M}$ admits a two-dimensional subgroup G_2 containing the cyclic symmetry and let X_0 and X_1 be a basis of the Lie algebra of G_2. In an adapted coordinate system the commutator relation of the Lie algebra of G_2,

$$[X_0\ X_1] = aX_0 + bX_2 \tag{1}$$

where a and b are constants, reduces to

$$\frac{\partial X_1^\mu}{\partial \phi} = bX_1^\mu \qquad \frac{\partial X_1^1}{\partial \phi} = a + bX_1^1$$

where Greek indices range over the values $2 \ldots n$. An elementary integration gives

$$\begin{array}{lll} X_1^\mu = B^\mu(x^\nu)\mathrm{e}^{b\phi} & X_1^1 = A(x^\nu)\mathrm{e}^{b\phi} + a/b & \text{for } b \neq 0 \\ X_1^\mu = B^\mu(x^\nu) & X_1^1 = A(x^\nu) + a\phi & \text{for } b = 0 \end{array}$$

where A and B^μ are arbitrary functions of integration. If X_1 is to be single-valued on $\mathcal{N}$, then these solutions must be periodic in ϕ with period 2π. This can only occur if $a = b = 0$ and so from Eq. (1) G_2 must be Abelian.

The dimensionality of the manifold, the existence of a metric and the fact that the transformation group is an isometry group are not used in the proof. Hence we have shown that any two-dimensional Lie transformation group which acts on an n-dimensional manifold $\mathcal{M}$ and which contains a one-dimensional subgroup with cicular orbits, must be Abelian. Thus *a fortiori* the result holds when the G_2 is group of motions, conformal motions, affine collineations or projective collineations. This remarkably simple and general

result has appeared in the literature previously (for example Bičák & Schmidt[4]) but is perhaps not widely known.

3 Cyclically Symmetric Manifolds Admitting a G_3

Suppose that $\mathcal{M}$ admits a three-dimensional Lie transformation group G_3 containing a one-dimensional subgroup acting cyclically on $\mathcal{M}$ generated by the vector field X_0. Let X_1 and X_2 be vector fields on $\mathcal{M}$ which, with X_0, form a basis of the Lie algebra of G_3. Now either this Lie algebra admits a two-dimensional subalgebra containing X_0 or there is no such subalgebra.

In the former case this subalgebra is Abelian by the result proved in the previous section. We may assume, without loss of generality, that X_0 and X_1 form a basis of this subalgebra and consequently the commutation relations involving X_0 can be written in the form

$$[X_0\ X_1] = 0 \qquad [X_0\ X_2] = aX_0 + bX_1 + cX_2$$

where a,b and c are constants. In a coordinate system adapted to X_0 in which $X_0 = \partial_\phi$ these equations become

$$\frac{\partial X_1^i}{\partial\phi} = 0 \qquad \frac{\partial X_2^i}{\partial\phi} = a\delta_1^i + bX_1^i + cX_2^i$$

On integrating these equations and using the fact that X_1 and X_2 must be periodic in ϕ with period 2π, we may deduce that $a = b = c = 0$. Thus X_0 commutes with both X_1 and X_2. The remaining basis freedom preserving X_0 is

$$\tilde{X}_1 = \alpha X_1 + \beta X_2 + \lambda X_0 \qquad \tilde{X}_2 = \gamma X_1 + \delta X_2 + \mu X_0$$

subject to the condition $\alpha\delta - \beta\gamma \neq 0$. Using this basis freedom we may reduce the commutators of the Lie algebra of G_3 to one of the following forms

$[X_0\ X_1] = 0$	$[X_0\ X_2] = 0$	$[X_1\ X_2] = 0$	Bianchi type I
$[X_0\ X_1] = 0$	$[X_0\ X_2] = 0$	$[X_1\ X_2] = X_0$	Bianchi type II
$[X_0\ X_1] = 0$	$[X_0\ X_2] = 0$	$[X_1\ X_2] = X_2$	Bianchi type III

These are the canonical forms for the commutators of Bianchi types I, II and III algebras given by Petrov[5] (apart from renumbering of the basis vectors for type III).

If the Lie algebra of G_3 has no two-dimensional subalgebra containing X_0, we may always choose basis vectors X_1 and X_2 such that the commutation relations involving X_0 become

$$[X_0\ X_1] = X_2 \qquad [X_0\ X_2] = aX_0 + bX_1 + cX_2$$

where a, b and c are constants. In terms of a coordinate system adapted to X_0 in which $X_0 = \partial_\phi$ these become

$$\frac{\partial X_1^i}{\partial \phi} = X_2^i \qquad \frac{\partial X_2^i}{\partial \phi} = a\delta_1^i + bX_1^i + cX_2^i$$

A straightforward integration of these equations reveals that, for solutions periodic in ϕ with period 2π, we must have $c = 0$ and $b = -n^2$ for some positive integer n. Then by a redefinition of the basis vector $\tilde{X}_1 = nX_1 - a/nX_0$, we can set $a = 0$. Hence the commutation relations become

$$[X_0\ X_1] = nX_2 \qquad [X_0\ X_2] = -nX_1 \qquad [X_1\ X_2] = dX_0 + eX_1 + fX_2$$

where d, e and f are constants. The Jacobi identity

$$[X_0\ [X_1\ X_2]] + [X_1\ [X_2\ X_0]] + [X_2\ [X_0\ X_1]] = 0$$

implies that

$$[X_0\ \ (dX_0 + eX_1 + fX_2)] = n(eX_2 - fX_1) = 0$$

Thus $e = f = 0$. Three algebraic distinct types arise: namely Bianchi types $\mathrm{VII}_q = 0$, VIII or IX depending on whether $d = 0,\ < 0,\ > 0$ respectively. The commutation relations may be written in one of the following forms

$[X_0\ X_1] = nX_2$	$[X_0\ X_2] = -nX_1$	$[X_1\ X_2] = 0$	Bianchi type $\mathrm{VII}_{q=0}$
$[X_0\ X_1] = nX_2$	$[X_0\ X_2] = -nX_1$	$[X_1\ X_2] = -X_0$	Bianchi type VIII
$[X_0\ X_1] = nX_2$	$[X_0\ X_2] = -nX_1$	$[X_1\ X_2] = X_0$	Bianchi type IX

where, in the last two types we have set $d = \mp 1$ by the rescaling

$$\tilde{X}_1 = 1/\sqrt{|d|}X_1 \qquad \tilde{X}_2 = 1/\sqrt{|d|}X_2$$

These commutators are closely related to the canonical forms of Bianchi types $\mathrm{VII}_{q=0}$, VIII and IX given by Petrov [5]. To get the canonical forms we would need to scale X_0 to set $n = 1$. However this cannot be done whilst preserving both the equation $X_0 = \partial_\phi$ and the 2π periodicity of the coordinate ϕ.

Only six of the nine Bianchi types can occur; Bianchi types IV, V and VI are excluded. Note also that canonical forms of the algebras of Bianchi types VI and VII depend on an arbitrary real parameter q and so each contain an infinite number of algebraically distinct cases; those in type VI are excluded completely and of those in type VII only a single case, $q = 0$, can occur. Moreover in all the Bianchi types that are permitted, the cyclic vector X_0 is aligned with a vector of a basis in which the commutation relations take their canonical form.

4 Cyclically symmetric manifolds admitting a G_{m+1}

Suppose now that $\mathcal{M}$ admits an $(m+1)$-dimensional Lie transformation group G_{m+1} containing a one-dimensional subgroup acting cyclically on $\mathcal{M}$ generated by the vector field X_0. Let X_a be vector fields on $\mathcal{M}$ which, with X_0, form a basis of the Lie algebra of G_{m+1}. Here and below indices a, b and c take values in the range $1 \ldots m$. The commutators involving X_0 may be written in the form

$$[X_0\ X_a] = C_a^b X_b + D_a X_0 \tag{2}$$

where C_a^b and D_a are constants.

If we introduce new basis vectors $\tilde{X}_a$ given by $\tilde{X}_a = P_a^b X_b$, the structure constants transform as follows

$$\tilde{\mathbf{C}} = \mathbf{PCP}^{-1} \qquad\qquad \tilde{\mathbf{D}} = \mathbf{PD}$$

where for simplicity we have used standard matrix notation. Using these transformations we can reduce $\mathbf{C}$ to Jordan normal form. In what follows we will work in a basis in which the structure constants C_a^b are in Jordan normal form but, for typographic simplicity tildes will be omitted.

In terms of a coordinate system adapted to X_0 in which $X_0 = \partial_\phi$ the commutation relations in Eq. (2) become

$$\frac{\partial X_a^\mu}{\partial \phi} = C_a^b X_b^\mu \qquad \frac{\partial X_a^1}{\partial \phi} = C_a^b X_b^1 + D_a \tag{3}$$

where Greek indices range over $2 \ldots n$. If the solutions of these equations are to be periodic in ϕ with period 2π, then the eigenvalues λ of $\mathbf{C}$ must either be zero or of the form $\lambda = \pm in$ where n is a positive integer. Moreover all of the Jordan blocks must be simple or equivalently the minimal polynomial of $\mathbf{C}$ must have no repeated factors.

Suppose without loss of generality that $\mathbf{C}$ has p $(0 \le 2p \le m)$ eigenvalues of the form in_j (n_j positive integers and $1 \le j \le p$) with corresponding complex eigenvectors $Z_j \equiv X_{2j} + iX_{2j-1}$ plus $m - 2p$ zero eigenvalues with corresponding real eigenvectors X_k $(2p+1 \le k \le m)$. Choosing these m (real) vectors X_a as the basis vectors, the commutators become

$$\begin{array}{lll} [X_0\ X_{2j-1}] = n_j X_{2j} & & \text{for } 1 \le j \le p \\ [X_0\ X_{2j}] \quad = -n_j X_{2j-1} & & \text{and } 0 \le 2p \le m \\ [X_0\ X_k] \quad\ \, = 0 & & \text{for } 2p+1 \le k \le m \end{array} \tag{4}$$

In the above commutators the structure constants D_a that appeared in Eq. (2) have been set to zero. This is valid since, for $2p+1 \le k \le m$, the vanishing of

D_k is a consequence of the periodicity of the solution of the second of Eqs. (3) which reduces to

$$\frac{\partial X_k^1}{\partial \phi} = D_k$$

For $1 \leq j \leq p$, D_{2j-1} and D_{2j} can be set to zero by a transformation of the basis vectors of the form

$$\tilde{X}_{2j-1} = X_{2j-1} - D_{2j}/n_j X_0 \qquad \tilde{X}_{2j} = X_{2j} + D_{2j-1}/n_j X_0$$

Thus even in the general case the existence of a one-dimensional group acting cyclically on the $\mathcal{M}$ imposes quite strong restrictions on the allowed form of the commutation relations involving X_0. Also the results of sections 2 and 3 can be seen to be special cases of the general result just proved.

For the G_4 case ($m = 3$) two classes of algebra arise with commutators

$$\begin{array}{lll} [X_0\ X_1] = 0 & [X_0\ X_2] = 0 & [X_0\ X_3] = 0 \\ [X_0\ X_1] = nX_2 & [X_0\ X_2] = -nX_1 & [X_0\ X_3] = 0 \end{array}$$

corresponding to the cases $p = 0$ and $p = 1$ in Eq. (4) respectively.

The Jacobi identities further restrict the structure constants appearing in the remaining commutators. In fact it is possible to enumerate completely the four-dimensional algebras of G_4 groups compatible with cyclic symmetry and to relate them to the eight types listed by Kruchkovich [6] and Petrov [5] in their complete classification of *all* four-dimensional Lie algebras. Examples of all eight of the types can occur, but in many cases only a zero-parameter or one-parameter subset of a two-parameter Kruchkovich-Petrov type is permitted and some subclasses of the Kruchkovich-Petrov types are excluded completely. Furthermore the vector X_0 generating the cyclic subgroup is always nicely aligned with the canonical bases used by Kruchkovich and Petrov. A complete account of the results for G_4 will appear elsewhere.

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